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ENDS OF GRAPHED EQUIVALENCE RELATIONS, II

BY

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ABSTRACT

Given a graphing $\mathscr{G}$ of a countable Borel equivalence relation on a Polish space, we show that if there is a Borel way of selecting a non-empty closed set of countably many ends from each $\mathscr{G}$-component, then there is a Borel way of selecting an end or line from each $\mathscr{G}$-component. Our method yields also Glimm–Effros style dichotomies which characterize the circumstances under which: (1) there is a Borel way of selecting a point or end from each $\mathscr{G}$-component; and (2) there is a Borel way of selecting a point, end or line from each $\mathscr{G}$-component.

1. Introduction

A topological space X is **Polish** if it is separable and completely metrizable. A Borel equivalence relation E on X is **countable** if all of its classes are countable. The descriptive set-theoretic study of such equivalence relations has blossomed over the last several years (see, for example, Jackson-Kechris-Louveau [2]). A Borel graph $\mathscr{G} \subseteq X \times X$ is a **graphing** of E if its connected components coincide with the equivalence classes of E.

A **ray** through $\mathscr{G}$ is an injective sequence $\alpha \in X^{\mathbb{N}}$ such that

$$\forall n \in \mathbb{N} \ ((\alpha(n), \alpha(n+1)) \in \mathscr{G}).$$

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We use $[\mathscr{G}]^\infty$ to denote the standard Borel space of all such rays. A graph $\mathscr{T}$ is a **forest** (or **acyclic**) if its connected components are trees. Although these trees are unrooted, we can nevertheless recover their branches as equivalence classes of the associated **tail equivalence relation** $\mathscr{E}_\mathscr{T}$ on $[\mathscr{T}]^\infty$, given by

$$\alpha \mathscr{E}_\mathscr{T} \beta \Leftrightarrow \exists i, j \in \mathbb{N}\, \forall k \in \mathbb{N}\ (\alpha(i+k) = \beta(j+k)).$$

Generalizing this to graphs, we obtain the relation $\mathscr{E}_\mathscr{G}$ of **end equivalence**. Two rays α, β through $\mathscr{G}|[x]_E$ are **end equivalent** if for every finite set $S \subseteq [x]_E$, there is a path from α to β through the graph $\mathscr{G}_{\hat{S}} = \{(y,z) \in \mathscr{G}|[x]_E : y, z \notin S\}$ on $[x]_E$. Equivalently, α, β are end equivalent if there is an infinite family $\{\gamma_n\}_{n\in\mathbb{N}}$ of pairwise vertex disjoint paths from α to β. An **end** of $\mathscr{G}$ is an equivalence class of $\mathscr{E}_\mathscr{G}$.

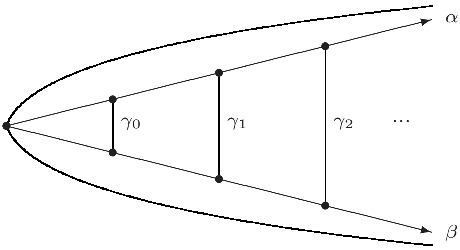

Figure 1. End-equivalent rays and the "infinite ladder" of paths between them.

In Miller [5], we characterized the equivalence relations which admit graphings for which there is a Borel way of selecting a given (finite) number of ends from each connected component. Here we characterize exactly when a given number of ends can be so chosen.

As the focus of Miller [5] was primarily on graphings whose components possess only finitely many ends, the topology on the space of ends did not come into play. Here it will be essential. The **topology on the space of ends** of $\mathscr{G}|[x]_E$ is that generated by the sets of the form

$$\mathscr{N}(\alpha, S) = \{\beta \in [\mathscr{G}|[x]_E]^\infty : \exists n \in \mathbb{N}\, \forall m \geq n\ (\alpha(m), \beta(m) \text{ are } \mathscr{G}_{\hat{S}}\text{-connected})\},$$

where $S \in [\mathscr{G}|[x]_E]^{<\infty}$ and $\alpha \in [\mathscr{G}|[x]_E]^\infty$. It is straightforward to check that this induces a zero-dimensional Polish topology on the ends of $\mathscr{G}|[x]_E$. When $\mathscr{G}|[x]_E$ is locally finite, it is even compact (we shall never make this assumption, however).

In §2, we describe a general method of building "combinatorially simple" Borel forests from a collection of data $(T, V, s_0, s_1, \ldots)$ which we call an **arboreal blueprint**. Here (T, V) is a finite tree and the sequence $(s_0, s_1, \ldots)$ encodes a way of recursively pasting together copies of (T, V) so as to obtain increasingly fine approximations to a Borel forest $\mathscr{T}$, which has the property that there is no Borel way of selecting a point or non-empty closed proper subset of ends from each component.

In §3, we introduce a notion of directability for graphings, which extends the corresponding notion for treeings (see §4 of Miller [5]). We show that a graphing is directable exactly when there is a Borel way of choosing a point or end from each component, and give a similar characterization of the circumstances under which there is a Borel way of choosing a point, end or line from each component.

In §4, we introduce tail-to-end embeddings of forests $\mathscr{T}$ into graphs $\mathscr{G}$ which, in particular, induce injections from the tail equivalence classes of $\mathscr{T}$ into the end equivalence classes of $\mathscr{G}$. We then show that tail-to-end embeddings behave nicely with respect to end selection.

In §5, we introduce a parameterized version of tail-to-end embedding, and describe the circumstances under which a finite graph can be so embedded into a graphing of a countable Borel equivalence relation.

In §6, we describe our main construction which, given an arboreal blueprint $(T, V, s_0, s_1, \ldots)$ with associated Borel forest $\mathscr{T}$, provides a way of building a tail-to-end embedding of $\mathscr{T}$ from a parameterized embedding of T.

In §7, we prove our main results. An arboreal blueprint $(T, V, s_0, s_1, \ldots)$ is **linear** if T is linear. Abusing notation slightly, we use $\mathscr{L}_0$ to denote the Borel forest associated with any linear arboreal blueprint, and we use $\mathscr{T}_0$ to denote the Borel forest associated with any non-linear arboreal blueprint. We show first the following two dichotomies.

Theorem A: *Suppose that $\mathscr{G}$ is a graphing of a countable Borel equivalence relation on a Polish space. Then exactly one of the following holds:*

1. *There is a Borel way of selecting a point or end from each $\mathscr{G}$-component.*
2. *There is a continuous tail-to-end embedding of $\mathscr{L}_0$ into $\mathscr{G}$.*

Theorem B: *Suppose that $\mathscr{G}$ is a graphing of a countable Borel equivalence relation on a Polish space. Then exactly one of the following holds:*

1. *There is a Borel way of selecting a point, end or line from each $\mathscr{G}$-component.*
2. *There is a continuous tail-to-end embedding of $\mathscr{T}_0$ into $\mathscr{G}$.*

The results of Miller [5] can be used to show that if there be a Borel way of selecting a non-empty set of finitely many ends from each $\mathscr{G}$-component, then there is a Borel way of selecting an end or line from each $\mathscr{G}$-component. Note that this conclusion is blatantly false if we merely ask that there is a Borel way of selecting a non-empty set of countably many ends from each $\mathscr{G}$-component. We close by proving the appropriate topological generalization:

Theorem C: *Suppose that X is a Polish space, E is a countable Borel equivalence relation, $\mathscr{G}$ is a graphing of E, and there is a Borel way of selecting a non-empty closed set of countably many ends from each $\mathscr{G}$-component. Then there is a Borel way of selecting an end or line from each $\mathscr{G}$-component.*

2. Examples

Here we describe a way of associating with each finite tree T a "combinatorially simple" Borel forest $\mathscr{T}$ with the property that there is no Borel way of selecting a point or non-empty closed proper subset of ends from each $\mathscr{T}$-component.

Throughout the paper, it will be convenient to identify elements of (finite or infinite) products $X_0 \times X_1 \times \cdots$ with the corresponding strings of the form $x(0)x(1)\ldots$, where $x(i) \in X_i$.

Suppose that T is a tree with finite vertex set V. The **boundary** of T is

$$\partial T = \{v \in V : v \text{ has at most one } T\text{-neighbor}\}.$$

For each $v_0 \in \partial T$, the **v_0-extension** of T is the tree T_{v_0} on $V \times 2$ given by

$$(v_1 i_1, v_2 i_2) \in T_{v_0} \Leftrightarrow ((v_1, v_2) \in T \text{ and } i_1 = i_2) \text{ or } (v_0 = v_1 = v_2 \text{ and } i_1 \neq i_2).$$

We also refer to T_{v_0} as a **one-step extension** of T.

An **arboreal blueprint** is a tuple $(T, V, s_0, s_1, \ldots)$, where V is a finite set of cardinality at least 2, T is a tree on V, $s_n \in \partial T \times 2^n$ and:

1. $\forall m < n \ (s_m \not\subseteq s_n)$.
2. $\forall s \in \partial T \times 2^{<\mathbb{N}} \ \exists n \in \mathbb{N} \ (s \subseteq s_n \text{ or } s_n \subseteq s)$.

Associated with each such blueprint is a family of trees T_n on $V \times 2^n$, which should be viewed as increasingly accurate approximations to a Borel forest $\mathscr{T}$

on $V \times 2^{\mathbb{N}}$. The tree T_0 is simply T, and T_{n+1} is defined recursively by $T_{n+1} = (T_n)_{s_n}$.

Letting F_n denote the equivalence relation on $V \times 2^{\mathbb{N}}$ which is given by

$$xF_n y \Leftrightarrow \forall m > n \ (x(m) = y(m)),$$

we then define $\mathscr{T}$ on $V \times 2^{\mathbb{N}}$ by

$$\mathscr{T} = \bigcup_{n \in \mathbb{N}} \{(x, y) \in V \times 2^{\mathbb{N}} : xF_n y \text{ and } (x|(n+1), y|(n+1)) \in T_n\},$$

where $x|(n+1) = x(0)x(1) \ldots x(n)$ and $y|(n+1) = y(0)y(1) \ldots y(n)$. Condition (1) ensures that the each point of $\partial T \times 2^{\mathbb{N}}$ has at most two $\mathscr{T}$-neighbors, and condition (2) ensures that the generic point of $\partial T \times 2^{\mathbb{N}}$ has at least two.

Despite the slightest of conflicts with the usual notation, we use E_0 to denote the equivalence relation on $V \times 2^{\mathbb{N}}$ given by

$$E_0 = \bigcup_{n \in \mathbb{N}} F_n = \{(x, y) \in V \times 2^{\mathbb{N}} : \exists n \in \mathbb{N} \, \forall m > n \ (x(m) = y(m))\}.$$

A **treeing** of an equivalence relation E is a graphing of E by a Borel forest.

Proposition 2.1: *$\mathscr{T}$ is a treeing of E_0.*

Proof. It is clear that $\mathscr{T}$ is a graphing of a subequivalence relation of E_0. To see that $\mathscr{T}$ is a graphing of E_0, suppose that xE_0y, and fix $n \in \mathbb{N}$ such that xF_ny. As $x|(n+1)$ and $y|(n+1)$ are T_n-connected, it follows from the definition of $\mathscr{T}$ that x and y are $\mathscr{T}$-connected.

It remains to check that $\mathscr{T}$ has no cycles. We must show that if $k \geq 2$ and $x_0, x_1, \ldots, x_k$ is an injective $\mathscr{T}$-path, then $(x_0, x_k) \notin \mathscr{T}$. Fix $n \in \mathbb{N}$ sufficiently large that $x_0 F_n x_1 F_n \cdots F_n x_k$. Then $x_0|(n+1), x_1|(n+1), \ldots, x_k|(n+1)$ is an injective T_n-path. As T_n is a tree, it follows that $(x_0|(n+1), x_k|(n+1)) \notin T_n$, thus $(x_0, x_k) \notin \mathscr{T}$. ∎

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\mathscr{G}$ is a graphing of E. We use $\sqcup$ to denote **disjoint union**. A **Borel way of selecting a point or closed proper subset of ends** from each $\mathscr{G}$-component is a Borel set $\mathscr{B} \subseteq X \sqcup [\mathscr{G}]^\infty$ such that for each $C \in X/E$, the intersection of $\mathscr{B}$ with $C \sqcup [\mathscr{G}|C]^\infty$ consists of either a single point of C or a non-empty closed $\mathscr{E}_{\mathscr{G}}$-invariant proper subset of $[\mathscr{G}|C]^\infty$.

Proposition 2.2: *There is no Borel way of selecting a point or closed proper subset of ends from each $\mathscr{T}$-component.*

Proof. Suppose, towards a contradiction, that $\mathscr{B} \subseteq (V \times 2^{\mathbb{N}}) \sqcup [\mathscr{T}]^\infty$ is a Borel set which consists of a point or non-empty $\mathscr{E}_{\mathscr{T}}$-invariant closed proper subset of ends from each $\mathscr{T}$-component. We draw out the desired contradiction by showing that $V \times 2^{\mathbb{N}}$ is the union of three meager sets. The first of these is given by

$$B_0 = \{x \in V \times 2^{\mathbb{N}} : \mathscr{B} \text{ selects a point from } [x]_{E_0}\}.$$

Given an equivalence relation E on X, the E**-saturation** of $B \subseteq X$ is given by

$$[B]_E = \{x \in X : \exists y \in B \ (xEy)\}.$$

Note that $B_0 = [\mathscr{B} \cap (V \times 2^{\mathbb{N}})]_{E_0}$.

Lemma 2.3: *B_0 is meager.*

Proof. Define $B = \mathscr{B} \cap (V \times 2^{\mathbb{N}})$ and suppose, towards a contradiction, that B_0 is non-meager. As E_0-saturation preserves meagerness, it follows that B is also non-meager. Given $s \in V \times 2^{<\mathbb{N}}$, we will use $\mathscr{N}_s$ to denote the set of $x \in V \times 2^{\mathbb{N}}$ such that $s \subseteq x$. As B is Borel, thus Baire measurable, it follows that there exists $s \in V \times 2^{<\mathbb{N}}$ such that B is comeager in $\mathscr{N}_s$. Then the set

$$C = (V \times 2^{\mathbb{N}}) \setminus [\mathscr{N}_s \setminus B]_{E_0}$$

is comeager, thus non-empty. As $\mathscr{N}_s \cap C \subseteq B \cap C$ and $\mathscr{N}_s$ intersects every E_0-class infinitely often, this contradicts the fact that B contains only one point from each equivalence class of $E_0|B_0$. ∎

The second set is given by

$$\begin{aligned} B_1 &= \{x \in V \times 2^{\mathbb{N}} : \mathscr{B} \text{ selects exactly one end from } \mathscr{T}|[x]_{E_0}\} \\ &= \{x \in (V \times 2^{\mathbb{N}}) \setminus B_0 : \forall \alpha, \beta \in \mathscr{B} \ (xE_0\alpha E_0\beta \Rightarrow \alpha \mathscr{E}_{\mathscr{T}} \beta)\}, \end{aligned}$$

where the notation $xE_0\alpha E_0\beta$ indicates that α and β are rays through $\mathscr{T}|[x]_{E_0}$.

Lemma 2.4: *B_1 is meager.*

Proof. Suppose, towards a contradiction, that B_1 is non-meager. As B_1 is E_0-invariant and $\mathbf{\Pi}^1_1$, thus Baire measurable, it follows that B_1 is comeager. Fix a comeager E_0-invariant Borel set $B \subseteq B_1$, and define $f : B \to B$ by letting $f(x)$ be the unique $\mathscr{T}$-neighbor of x which lies along a ray in $\mathscr{B}$ that originates at x. Then graph(f) is $\mathbf{\Sigma^1_1}$, thus f is Borel. Note also that $\mathscr{T}|B =$ graph$(f|B) \cup$ graph$(f^{-1}|B)$.

The **graph metric** associated with $\mathscr{T}$ is given by

$$d_{\mathscr{T}}(x,y) = \begin{cases} n & \text{if there is an injective } \mathscr{T}\text{-path from } x \text{ to } y \text{ of length } n, \\ \infty & \text{if } x, y \text{ are not } \mathscr{T}\text{-connected.} \end{cases}$$

Sublemma 2.5: $\forall x, y \in B \; (d_{\mathscr{T}}(x,y) \geq d_{\mathscr{T}}(f(x), f(y)))$.

Proof. Suppose that $d_{\mathscr{T}}(x,y) = n$, and let $z_0, z_1, \ldots, z_n$ be the injective $\mathscr{T}$-path from x to y. If $f(z_0) = z_1$, then it is clear that $d_{\mathscr{T}}(f(x), f(y)) \leq n$. Otherwise, the obvious induction shows that $\forall i < n \; (f(z_{i+1}) = z_i)$, thus $d_{\mathscr{T}}(f(x), f(y)) \leq n$. ∎

Note that each $x \in B \cap (\partial T \times 2^{\mathbb{N}})$ has a unique $\mathscr{T}$-neighbor $y \in B$ such that $x(0) \neq y(0)$. As the points of $\partial T \times 2^{\mathbb{N}}$ each have at most two $\mathscr{T}$-neighbors, it follows that the set $A = \{x \in B \cap (\partial T \times 2^{\mathbb{N}}) : x(0) \neq [f(x)](0)\}$ is a **complete section** for $E_0|B$ (i.e., $B = [A]_{E_0|B}$), thus non-meager. Putting

$$A_{v,w} = \{x \in B : x(0) = v \text{ and } [f(x)](0) = w\},$$

it follows that we can find $v \in \partial T$ and $w \neq v$ in V such that $A_{v,w}$ is non-meager.

Fix $s \in 2^{<\mathbb{N}}$ such that $A_{v,w}$ is comeager in $\mathscr{N}_{vs}$. Then the set

$$C = B \setminus [\mathscr{N}_{vs} \setminus A_{v,w}]_{E_0}$$

is comeager and $\mathscr{N}_{vs} \cap C \subseteq A_{v,w} \cap C$. Put $k = |s|$, and find $t \in \partial T_k$ such that there is a T_k-path of the form $ws, vs, \ldots, t$. As $t \in \partial T_k$, there exists $n \in \mathbb{N}$ such that $t \subseteq s_n$. It follows that there exists $u \in 2^{n-k}$ and a T_{n+1}-path of the form

$$wsu0, vsu0, \ldots, s_n0, s_n1, \ldots, vsu1, wsu1.$$

Fix $x \in 2^{\mathbb{N}}$ such that $vsu0x \in C$, and observe that

$$d_{\mathscr{T}}(vsu0x, vsu1x) < d_{\mathscr{T}}(wsu0x, wsu1x) = d_{\mathscr{T}}(f(vsu0x), f(vsu1x)),$$

which contradicts Sublemma 2.5. ∎

The final set is given by

$$\begin{aligned} B_2 &= \{x \in V \times 2^{\mathbb{N}} : \mathscr{B} \text{ selects at least two ends from } \mathscr{T}|[x]_{E_0}\} \\ &= \{x \in V \times 2^{\mathbb{N}} : \exists \alpha, \beta \in \mathscr{B} \; (x E_0 \alpha E_0 \beta \text{ and } (\alpha, \beta) \notin \mathscr{E}_{\mathscr{T}})\}. \end{aligned}$$

It now remains only to check the following

LEMMA 2.6: *B_2 is meager.*

Proof. We say that z is **$\mathscr{T}$-between** x and y if the injective $\mathscr{T}$-path from x to y goes through z, and we say that $B \subseteq X$ is **$\mathscr{T}$-convex** if

$$\forall x, y \in B \, \forall z \in X \ (z \text{ is } \mathscr{T}\text{-between } x \text{ and } y \Rightarrow z \in B).$$

Suppose, towards a contradiction, that B_2 is non-meager, and define $B \subseteq B_2$ by

$$B = \{x \in B_2 : \exists \alpha, \beta \in \mathscr{B} \ (\alpha(0) = \beta(0) = x \text{ and } \alpha(1) \neq \beta(1))\} .$$

It is clear that B is $\mathscr{T}$-convex. After throwing out an E_0-invariant meager Borel set, we can assume that both B and B_2 are Borel. As B is a complete section for $E_0|B_2$, it follows that B is non-meager. As $\mathscr{B}$ selects a proper closed subset of ends from each $\mathscr{T}$-component, it follows that B misses a point of every E_0-class, thus B is not comeager, so there exist $s, t \in 2^{<\mathbb{N}}$ such that B is comeager in $\mathscr{N}_s$ and meager in $\mathscr{N}_t$. By extending the longer of the two, we may assume that $|s| = |t|$. Set $C = B \setminus ([\mathscr{N}_s \setminus B]_{E_0} \cup [\mathscr{N}_t \cap B]_{E_0})$, noting that

$$(\dagger) \qquad \mathscr{N}_s \cap C \subseteq B \cap C \quad \text{and} \quad B \cap C \cap \mathscr{N}_t = \emptyset.$$

Put $k = |s| - 1 = |t| - 1$ and find $u \in \partial T_k$ such that t is T_k-between s and u. As $u \in \partial T_k$, there exists $n \in \mathbb{N}$ such that $u \subseteq s_n$. It then follows that there exists $s', t' \in 2^{n-k}$ and a T_{n+1}-path of the form

$$ss'0, \ldots, tt'0, \ldots, s_n0, s_n1, \ldots, tt'1, \ldots ss'1.$$

Fix $x \in 2^{\mathbb{N}}$ such that $ss'0x \in C$, and observe that $tt'0x$ is $\mathscr{T}$-between $ss'0x$ and $ss'1x$, thus $tt'0x \in B \cap C \cap \mathscr{N}_t$, which is the desired contradiction with $(\dagger)$. ∎

This ends the proof of Proposition 2.2. ∎

3. Directability

Here we introduce a notion of directability for graphings which characterizes the ability to select, in a Borel fashion, a point or end from each component. Similarly, we characterize the ability to select, in a Borel fashion, a point, end or line from each component.

We use $[\mathscr{G}]^{<\infty}$ to denote the standard Borel space of finite $\mathscr{G}$-connected subsets of X. For each $S \in [\mathscr{G}]^{<\infty}$, we use

$$\mathscr{G}_{\hat{S}} = \{(x, y) \in \mathscr{G} : x, y \in [S]_E \setminus S\}$$

to denote the graph on $[S]_E$ which is obtained from $\mathscr{G}|[S]_E$ by removing every edge that touches an element of S, and we use $E_{\hat{S}}$ to denote the equivalence relation on $[S]_E$ whose classes coincide with the connected components of $\mathscr{G}_{\hat{S}}$.

Let $[\mathscr{G}]^{\rightarrow}$ denote the standard Borel space of pairs of the form (S, C), where C is a connected component of $\mathscr{G}_{\hat{S}}$. Intuitively, we think of each pair $(S, C) \in [\mathscr{G}]^{\rightarrow}$ as indicating a preference that points of S should "flow towards C." We say that $(S, C), (T, D) \in [\mathscr{G}]^{\rightarrow}$ are **compatible** if either S and T lie in different E-classes or $C \cap D \neq \emptyset$, and we say that a set $\Phi \subseteq [\mathscr{G}]^{\rightarrow}$ is **directed** if all pairs $(S, C), (T, D) \in \Phi$ are compatible. This easily implies that Φ is the graph of a partial function. From this point forward, we will identify such sets with the corresponding partial function. We say that $\mathscr{S} \subseteq [\mathscr{G}]^{<\infty}$ is **directable** if there is a directed Borel set $\Phi \subseteq [\mathscr{G}]^{\rightarrow}$ such that $\mathrm{dom}(\Phi) = \mathscr{S}$, and $\mathscr{G}$ is **directable** if $[\mathscr{G}]^{<\infty}$ is directable. This generalizes the notion of directability for forests from §4 of Miller [5]:

Proposition 3.1: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\mathscr{T}$ is a treeing of E. Then the following are equivalent:*

1. *There is a directed Borel set $\Phi \subseteq [\mathscr{T}]^{\rightarrow}$ such that* $\mathrm{dom}(\Phi) = [\mathscr{T}]^{<\infty}$.
2. *There is a Borel function $f : X \rightarrow X$ such that $\mathscr{T} = \mathrm{graph}(f) \cup \mathrm{graph}(f^{-1})$.*

Proof. To see (1) $\Rightarrow$ (2), suppose that $\Phi \subseteq [\mathscr{T}]^{\rightarrow}$ is a directed Borel set of full domain, and define $f : X \rightarrow X$ by

$$f(x) = \text{ the unique element of } (\{x\} \cup \mathscr{T}_x) \cap \Phi(\{x\}).$$

To see that $\mathscr{T} = \mathrm{graph}(f) \cup \mathrm{graph}(f^{-1})$, simply observe that if $(x, y) \in \mathscr{T}$, then the fact that $\Phi(\{x\}) \cap \Phi(\{y\}) \neq \emptyset$ that $y \in \Phi(\{x\})$ or $x \in \Phi(\{y\})$, thus $f(x) = y$ or $f(y) = x$.

To see (2) $\Rightarrow$ (1), suppose that $f : X \rightarrow X$ is a Borel function such that $\mathscr{T} = \mathrm{graph}(f) \cup \mathrm{graph}(f^{-1})$, and note that if $S \subseteq [x]_E$, then the forward orbit $x, f(x), \ldots$ eventually settles into a single connected component C of $\mathscr{T}_{\hat{S}}$. Moreover, this connected component is independent of the choice of x, since for any $y \in [x]_E$, the sequences $x, f(x), \ldots$ and $y, f(y), \ldots$ are tail-equivalent. Set $\Phi(S) = C$. To see that Φ is directed, simply note that for all $x \in X$ and $S, T \in [\mathscr{G}|[x]_E]^{<\infty}$, there exists $n \in \mathbb{N}$ sufficiently large that $f^n(x) \in \Phi(S) \cap \Phi(T)$, thus $\Phi(S) \cap \Phi(T) \neq \emptyset$. ∎

The following criterion for directability will be useful in the upcoming sections.

Proposition 3.2: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a graphing of E, and there are countably many directed Borel sets whose domains cover $[\mathscr{G}]^{<\infty}$. Then $\mathscr{G}$ is directable.*

Proof. The main observation is the following:

Lemma 3.3: *Suppose that $\Phi_1, \Phi_2 \subseteq [\mathscr{G}]^{\rightarrow}$ are directed Borel sets. Then there is an E-invariant Borel set $B \subseteq X$ and a directed Borel set $\Phi \subseteq [\mathscr{G}|B]^{\rightarrow}$ such that $E|(X \setminus B)$ is smooth, $\Phi_1|B \subseteq \Phi$, and $\mathrm{dom}(\Phi_2|B) \subseteq \mathrm{dom}(\Phi)$.*

Proof. Let Ψ denote the set of all pairs $(S_2, C_2) \in \Phi_2$ which are compatible with every element of Φ_1. Clearly the set $\Phi_1 \cup \Psi$ is directed. We say that a pair $(S_2, C_2) \in \Phi_2$ is **good** if there are $(S_1, C_1), (T_1, D_1) \in \Phi_1$, $(T_2, D_2) \in \Phi_2$, and $S, T \in [\mathscr{G}]^{<\infty}$ with $S_1 \cup S_2 \subseteq S$, $T_1 \cup T_2 \subseteq T$, $S \cap T = C_1 \cap C_2 = D_1 \cap D_2 = \emptyset$, and $S_2 \subseteq D_2$. While this implies that $S_2 \notin \mathrm{dom}(\Psi)$, it ensures that $D_1 \cap S_2 \subseteq D_1 \cap D_2 = \emptyset$, so that every point of D_1 is $E_{\hat{S}_2}$-related to T_1, thus $D_1 \subseteq [T_1]_{E_{\hat{S}_2}}$. It follows that we can safely change the component associated with S_2 from C_2 to $[T_1]_{E_{\hat{S}_2}}$.

By the Lusin–Novikov uniformization theorem (see for example, §18 of Kechris [3]), there is a Borel function $(S_2, C_2) \mapsto ((S_1, C_1), (T_1, D_1), (T_2, D_2), S, T)$ which assigns witnesses to good pairs. Let Ψ' denote the corresponding set of pairs of the form $(S_2, [T_1]_{E_{\hat{S}_2}})$. Clearly the set $\Phi_1 \cup \Psi \cup \Psi'$ is directed. Put $\mathscr{S} = \mathrm{dom}(\Phi_2) \setminus (\mathrm{dom}(\Psi) \cup \mathrm{dom}(\Psi'))$. It only remains to check that the restriction of E to the set $A = \bigcup \mathscr{S}$ is smooth.

By Proposition 7.3 of Kechris–Miller [4], there is a Borel complete section $D \subseteq A$ for $E|A$ and a finite Borel equivalence relation $F \subseteq E$ on D such that every F-class is $\mathscr{G}$-connected and contains incompatible pairs $(S_1, C_1) \in \Phi_1, (S_2, C_2) \in \Phi_2$, where (S_2, C_2) is not good. It then follows from the directedness of Φ_2 that every $(E|A)$-class contains exactly one F-class, thus $E|A$ is smooth, and the lemma follows. ∎

Now fix countably many directed sets $\Phi_0, \Phi_1, \ldots$ whose domains cover $[\mathscr{G}]^{<\infty}$, and repeatedly apply the lemma to find an E-invariant Borel set $B \subseteq X$ such that $E|(X \setminus B)$ is smooth, as well as Borel sets $\Psi_0 \subseteq \Psi_1 \subseteq \cdots$ such that $\Psi = \bigcup_{n \in \mathbb{N}} \Psi_n$ is directed and $\mathrm{dom}(\Phi_n|B) \subseteq \mathrm{dom}(\Psi_n)$. As every graphing of a smooth countable Borel equivalence relation is trivially directable, the proposition follows. ∎

Let $\mathscr{I}$ denote the σ-ideal of directable Borel subsets of $[\mathscr{G}]^{<\infty}$. A **Borel way of selecting a point or end** from each $\mathscr{G}$-component is a Borel set $\mathscr{B} \subseteq X \sqcup [\mathscr{G}]^{\infty}$ such that for each $C \in X/E$, the intersection of $\mathscr{B}$ with $C \sqcup [\mathscr{G}|C]^{\infty}$ consists of either a single point of C or a single equivalence class of $\mathscr{E}_{\mathscr{G}|C}$.

Proposition 3.4: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\mathscr{G}$ is a graphing of E. Then the following are equivalent:*

1. $[\mathscr{G}]^{<\infty} \in \mathscr{I}$.
2. *There is a Borel way of selecting a point or end from each $\mathscr{G}$-component.*

Proof. To see (1) ⇒ (2), fix a directed Borel set $\Phi \subseteq [\mathscr{G}]^{\rightarrow}$ of full domain. As the set $\{x \in X : x \in \Phi(\{x\})\}$ is a Borel partial transversal of E, we can assume that $\Phi(\{x\})$ never includes x. A ray α through $\mathscr{G}|[x]_E$ is **compatible** with Φ if

$$\forall S \in [\mathscr{G}|[x]_E]^{<\infty}\, \exists n \in \mathbb{N}\, \forall m \geq n\ (\alpha(m) \in \Phi(S)).$$

It is clear that the set $\mathscr{B}$ of rays compatible with Φ is Borel and $\mathscr{E}_{\mathscr{G}}$-invariant, and a simple induction shows that there is a ray through every connected component of $\mathscr{G}$ which is compatible with Φ. As any two such rays in the same E-class are necessarily end equivalent, it follows that $\mathscr{B}$ selects an end from each $\mathscr{G}$-component.

To see (2) ⇒ (1), fix a Borel set $\mathscr{B} \subseteq X \sqcup [\mathscr{G}]^{\infty}$ which consists of either a point or end from each $\mathscr{G}$-component. As $E|[\mathscr{B} \cap X]_E$ is smooth, we can assume that $\mathscr{B} \subseteq [\mathscr{G}]^{\infty}$. For each $S \in [\mathscr{G}]^{<\infty}$, let $\mathscr{B}_{\hat{S}}$ denote the set of rays in $\mathscr{B}$ through $[S]_E \setminus S$, and set

$$\Phi(S) = \{x \in X : \forall \alpha \in \mathscr{B}_{\hat{S}}\ (x E_{\hat{S}} \alpha(0))\}.$$

Then $\Phi(S) = \{x \in X : \exists \alpha \in \mathscr{B}_{\hat{S}}\ (x E_{\hat{S}} \alpha(0))\}$, thus Φ is both $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_1$, and hence Borel. Moreover, it is clear that if $S, T \in [\mathscr{G}]^{<\infty}$ lie in the same E-class, then $\Phi(S) \cap \Phi(T)$ contains a ray in $\mathscr{B}$, and is therefore non-empty. It follows that Φ is directed, thus $\mathscr{G}$ is directable. ∎

We say that a set $\mathscr{S} \subseteq [\mathscr{G}]^{<\infty}$ is **non-linear** if there are pairwise disjoint sets $S \in [\mathscr{G}]^{<\infty}$ and $S_1, S_2, S_3 \subseteq [S]_E$ in $\mathscr{S}$ such that $[S_1]_{E_{\hat{S}}}, [S_2]_{E_{\hat{S}}}, [S_3]_{E_{\hat{S}}}$ are pairwise disjoint. We use $\mathscr{J}$ to denote the family of subsets of $[\mathscr{G}]^{<\infty}$ which are contained in the union of a directable Borel set and a linear Borel set. A **Borel way of selecting a point, end or line** from each $\mathscr{G}$-component is a Borel set $\mathscr{B} \subseteq X \sqcup [\mathscr{G}]^\infty$ such that for each equivalence class C of E, the intersection of $\mathscr{B}$ with $C \sqcup [\mathscr{G}|C]^\infty$ consists of either a single point of C, a single equivalence class of $\mathscr{E}_{\mathscr{G}|C}$, or points $x_n \in C$, for $n \in \mathbb{Z}$, such that $(x_m, x_n) \in \mathscr{G} \Leftrightarrow |m-n| = 1$.

Proposition 3.5: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\mathscr{G}$ is a graphing of E. Then the following are equivalent:*

1. $[\mathscr{G}]^{<\infty} \in \mathscr{J}$.
2. *There is a Borel way of selecting a point, end or line from each $\mathscr{G}$-component.*

Proof. To see (1) $\Rightarrow$ (2), suppose that $[\mathscr{G}]^{<\infty}$ is contained in the union of a directable Borel set $\mathscr{S}_1 \subseteq [\mathscr{G}]^{<\infty}$ and a linear Borel set $\mathscr{S}_2 \subseteq [\mathscr{G}]^{<\infty}$. By Sublemma 5.4 of Miller [5], there are Borel sets $\mathscr{S}'_n$ such that each $\mathscr{S}'_n$ is pairwise disjoint and $\mathscr{S}_2 = \bigcup_{n\in\mathbb{N}} \mathscr{S}'_n$. Given $C \in X/E$, $S \in [\mathscr{G}|C]^{<\infty}$, and $\alpha \in [\mathscr{G}|C]^\infty$, let $C(\alpha, S)$ denote the $\mathscr{G}_{\hat{S}}$-component such that $\alpha(i) \in C(\alpha, S)$, for i sufficiently large. We say that α is **inseparable** from $\mathscr{S}'_n$ if

$$\forall S \in [\mathscr{G}|C]^{<\infty}\ (C(\alpha, S) \cap \bigcup \mathscr{S}'_n \neq \emptyset).$$

Let $\mathscr{B}_n$ denote the set of rays which are inseparable from $\mathscr{S}'_n$, and set

$$B_n = \{x \in X : \mathscr{B}_n \cap [\mathscr{G}|[x]_E]^\infty \neq \emptyset\}.$$

It follows from the linearity of $\mathscr{S}'_n$ that $\mathscr{B}_n$ contains at most 2 ends from each equivalence class of E, thus B_n is Borel and Theorems 2.1 and 5.1 of Miller [5] imply that there is a Borel way of selecting a point, end or line from each component of $\mathscr{G}|[B_n]_E$. It then follows from Proposition 3.4 that there is a Borel way of selecting a point, end or line from each $\mathscr{G}$-component.

To see (2) $\Rightarrow$ (1), it is enough to show that if $\mathscr{B} \subseteq [\mathscr{G}]^{<\infty}$ selects one or two ends from each $\mathscr{G}$-component, then $[\mathscr{G}]^{<\infty} \in \mathscr{J}$. For each $i \in \{1, 2\}$, let $\mathscr{S}_i$ be the set of $S \in [\mathscr{G}]^{<\infty}$ such that there are exactly i equivalence classes of $E_{\hat{S}}$ of the form $C(\alpha, S)$, where $\alpha \in \mathscr{B}$. Proposition 6.1 of Miller [5] ensures that $\mathscr{S}_i$ is Borel, and it is easily verified that $\mathscr{S}_1$ is directable and $\mathscr{S}_2$ is linear, thus $[\mathscr{G}]^{<\infty} \in \mathscr{J}$. ∎

4. Tail-to-end embeddings

Here we introduce the notion of tail-to-end embedding and show that it behaves nicely with respect to end selection.

Suppose that E is a countable Borel equivalence relation on X and $\mathscr{G}$ is a graphing of E. We use $\mathscr{E}$ to denote the equivalence relation on $[\mathscr{G}]^{<\infty}$ given by

$$S \mathscr{E} T \Leftrightarrow \exists x \in X \ (S, T \subseteq [x]_E).$$

Given a Borel set $\mathscr{S} \subseteq [\mathscr{G}]^{<\infty}$, the **induced graph** on $\mathscr{S}$ is the graphing of $\mathscr{E}|\mathscr{S}$ which consists of the pairs (S,T) of distinct elements of $\mathscr{S}$ for which there is a $\mathscr{G}$-path from S to T which avoids the rest of $\mathscr{S}$.

Now suppose that $\mathscr{T}$ is a Borel forest on Y. A **tail-to-end embedding** of $\mathscr{T}$ into $\mathscr{G}$ is a Borel injection $\pi : Y \to [\mathscr{G}]^{<\infty}$ such that $\mathscr{S} = \pi(Y)$ is pairwise disjoint and

$$\forall y_1, y_2 \in Y \ ((y_1, y_2) \in \mathscr{T} \Leftrightarrow (\pi(y_1), \pi(y_2)) \in \mathscr{G}_{\mathscr{S}}).$$

For $\kappa \leq \aleph_0$, a **Borel way of selecting a point or non-empty closed set of $\leq \kappa$ ends** from each $\mathscr{G}$-component is a Borel set $\mathscr{B} \subseteq X \sqcup [\mathscr{G}]^{\infty}$ such that for each $C \in X/E$, the intersection of $\mathscr{B}$ with $C \sqcup [\mathscr{G}|C]^{\infty}$ consists of either a point of C or a non-empty $\mathscr{E}_{\mathscr{G}}$-invariant closed set of $\leq \kappa$ ends.

Proposition 4.1: *Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, $\mathscr{G}$ is a graphing of E, $\mathscr{T}$ is a treeing of F, there is a Borel way of selecting a point or non-empty closed set of $\leq \kappa$ ends from each $\mathscr{G}$-component and $\mathscr{T}$ tail-to-end embeds into $\mathscr{G}$. Then there is a Borel way of selecting a point or non-empty closed set of $\leq \kappa$ ends from each $\mathscr{T}$-component.*

Proof. Fix a Borel set $\mathscr{B} \subseteq X \sqcup [\mathscr{G}]^{\infty}$ which selects a point or non-empty $\mathscr{E}_{\mathscr{G}}$-invariant closed set of $\leq \kappa$ ends from each $\mathscr{G}$-component, as well as a tail-to-end embedding $\pi : Y \to [\mathscr{G}]^{<\infty}$ of $\mathscr{T}$ into $\mathscr{G}$ with range $\mathscr{S} = \pi(Y)$. Set $Z = \{y \in Y : |[y]_E| \geq 2\}$. As π is an embedding of $F|Z$ into $\mathscr{E}$, we can assume that $\mathscr{B} \subseteq [\mathscr{G}]^{\infty}$. It will also be convenient to assume that $\mathscr{S}$ is an $\mathscr{E}$-complete section.

Let $\mathscr{B}_{\mathscr{S}}$ denote the set of rays in $\mathscr{B}$ which are inseparable from $\mathscr{S}$. Then $\mathscr{B}_{\mathscr{S}}$ selects an $\mathscr{E}_{\mathscr{G}}$-invariant closed set of ends from each $\mathscr{G}$-component, and the

Lusin-Novikov uniformization theorem ensures that $\mathscr{B}_{\mathscr{S}}$ is Borel. Set

$$A = \{x \in X : \mathscr{B}_{\mathscr{S}} \cap [\mathscr{G}|[x]_E]^\infty \neq \emptyset\}.$$

Lemma 4.2: *A is Borel.*

Proof. By Proposition 6.1 of Miller [5], there is a Borel $\mathscr{E}_{\mathscr{G}}$-complete section $\mathscr{A} \subseteq [\mathscr{G}]^\infty$ such that $\mathscr{E}_{\mathscr{G}}|\mathscr{A}$ is countable. Noting that

$$A = \{x \in X : \mathscr{A} \cap \mathscr{B}_{\mathscr{S}} \cap [\mathscr{G}|[x]_E]^\infty \neq \emptyset\},$$

the lemma follows from the fact that images of Borel sets under countable-to-one Borel functions are themselves Borel (see, for example, §18 of Kechris [3]). ∎

Next, we deal with the complement of the set $B = \pi^{-1}([\mathscr{G}|A]^{<\infty})$.

Lemma 4.3: *$F|(Y \setminus B)$ is smooth.*

Proof. As π is an embedding of $F|Z$ into $\mathscr{E}$, it is enough to show that $E|(X \setminus A)$ is smooth. Let $\mathscr{S}'$ denote the set of $S' \subseteq X \setminus A$ in $\mathscr{S}$ for which there exists $\alpha \in \mathscr{B}$ which goes through S' but avoids the rest of $\mathscr{S}$.

Sublemma 4.4: *$\mathscr{S}'$ is Borel.*

Proof. By Proposition 6.1 of Miller [5], there is a Borel $\mathscr{E}_{\mathscr{G}}$-complete section $\mathscr{A} \subseteq [\mathscr{G}]^\infty$ such that $\mathscr{E}_{\mathscr{G}}|\mathscr{A}$ is countable. We can clearly assume that $\mathscr{A}$ is closed under tail-equivalence. It follows that $\mathscr{S}'$ is the set of $S' \in \mathscr{S}$ for which there is a ray $\alpha \in \mathscr{A} \cap \mathscr{B}$ which goes through S' but avoids the rest of $\bigcup \mathscr{S}$. As images of Borel sets under countable-to-one Borel functions are Borel, so too is $\mathscr{S}'$. ∎

By Proposition 2.1 of Miller [5], it is enough to show that no ray of $\mathscr{G}|(X \setminus A)$ goes through infinitely many points of $\bigcup \mathscr{S}'$. Suppose, towards a contradiction, that $\alpha \in [\mathscr{G}|(X \setminus A)]^\infty$ goes through infinitely many points of $\bigcup \mathscr{S}'$. Of course, this implies that α is inseparable from $\mathscr{S}$. Fix distinct $S_n \in \mathscr{S}'$ and $\alpha_n \in \mathscr{B}$ such that α and α_n go through S_n, and α_n avoids the rest of $\mathscr{S}$.

Sublemma 4.5: *For all $n \in \mathbb{N}$, there is at most one $m \neq n$ such that α_m and α_n have a point in common.*

Proof. Suppose, towards a contradiction, that there exist $\ell < m < n$ such that any two of $\alpha_l, \alpha_m, \alpha_n$ have a point in common. Then there are $\mathscr{G}$-paths between any two of S_ℓ, S_m, S_n which avoid the rest of $\mathscr{S}$, thus S_ℓ, S_m, S_n form a 3-cycle in $\mathscr{G}_{\mathscr{S}}$, so $\pi^{-1}(S_\ell), \pi^{-1}(S_m), \pi^{-1}(S_n)$ form a 3-cycle in $\mathscr{T}$, which contradicts the fact that $\mathscr{T}$ is a forest. ∎

It now follows that for all $S \in [\mathscr{G}]^{<\infty}$, there exists $n \in \mathbb{N}$ such that S_n and α_n avoid S, thus α is in the closure of the ends selected by $\mathscr{B}$, so $\alpha \in \mathscr{B}_{\mathscr{S}}$, which contradicts the definition of A. ∎

It only remains to show that there is a Borel way of selecting $\leq \kappa$ ends from each component of $\mathscr{T}|B$. We say that a ray $\alpha \in [\mathscr{T}]^\infty$ **induces** a ray $\beta \in [\mathscr{G}]^\infty$ if β is inseparable from the set $\{\pi(\alpha(n))\}_{n\in\mathbb{N}}$.

Lemma 4.6: *Every ray of $\mathscr{T}$ induces a ray of $\mathscr{G}$.*

Proof. Set $S_n = \pi(\alpha(n))$, fix $\mathscr{G}$-paths $\gamma_{n,n+1}$ from S_n to S_{n+1} of minimal length, and let γ_{n+1} be an injective $\mathscr{G}$-path through S_{n+1} from the terminal point of $\gamma_{n,n+1}$ to the initial point of $\gamma_{n+1,n+2}$. As $\mathscr{T}$ is a treeing and π is a tail-to-end embedding, it follows that S_n and S_{n+2} lie in distinct components of $\mathscr{G}_{\hat{S}_{n+1}}$, thus $\gamma_{0,1}\gamma_1\gamma_{1,2}\gamma_2\ldots$ is a ray through $\mathscr{G}$, and it is clearly induced by $\mathscr{T}$. ∎

Let $\mathscr{A} \subseteq [\mathscr{T}]^\infty$ denote the set of rays of $\mathscr{T}$ which induce rays of $\mathscr{G}$ in $\mathscr{B}_{\mathscr{S}}$. Then Proposition 6.1 of Miller [5] ensures that $\mathscr{A}$ is a Borel $\mathscr{E}_{\mathscr{T}}$-invariant set which selects a non-empty closed set of $\leq \kappa$ ends from each component of $\mathscr{T}|B$. ∎

5. Parameterized embeddings

Here we discuss a parameterized notion of tail-to-end embedding.

We begin by fixing a variety of objects which will be of use throughout the rest of the paper. By Theorem 1 of Feldman–Moore [1], there is a countable group Γ of Borel automorphisms of $[\mathscr{G}]^{<\infty}$ such that $\mathscr{E} = \bigcup_{\gamma\in\Gamma} \mathrm{graph}(\gamma)$. Given a finite set $\Delta \subseteq \Gamma$ and $\delta \in \Delta$, we say that disjoint $\mathscr{E}$-related sets $S, S' \in [\mathscr{G}]^{<\infty}$ are (Δ, δ)**-linkable** if every path from $\Delta \cdot S$ to $\Delta \cdot S'$ goes through $\delta \cdot S$ and $\delta \cdot S'$. We use $\mathscr{I}_\Delta$ to denote the σ-ideal generated by Borel sets $\mathscr{S} \subseteq [\mathscr{G}]^{<\infty}$ such that $\delta(\mathscr{S}) \in \mathscr{I}$, for some $\delta \in \Delta$.

Suppose now that (T, V) is a finite tree. A **parameterized embedding** of T into $\mathscr{G}$ is a triple $(\Delta, \pi, \mathscr{S})$, where $\Delta \subseteq \Gamma$, $\pi : V \to \Delta$ is bijective, $\mathscr{S} \subseteq [\mathscr{G}]^{<\infty}$ is an $\mathscr{I}_{\pi(\partial T)}$-positive Borel set, and for every $S \in \mathscr{S}$, the map $v \mapsto \pi(v) \cdot S$ is a tail-to-end embedding.

PROPOSITION 5.1: *Suppose that there is no Borel way of selecting a point or end from each $\mathscr{G}$-component. Then there is a parameterized embedding of the tree on two points into $\mathscr{G}$.*

Proof. For each $\gamma \in \Gamma$, set $\Delta_\gamma = \{1_\Gamma, \gamma\}$ and $\mathscr{S}_\gamma = \{S \in [\mathscr{G}]^{<\infty} : S \cap \gamma \cdot S = \emptyset\}$.

LEMMA 5.2: *There exists $\gamma \in \Gamma$ such that $\mathscr{S}_\gamma \notin \mathscr{I}_{\Delta_\gamma}$.*

Proof. Suppose, towards a contradiction, that each $\mathscr{S}_\gamma$ is $\mathscr{I}_{\Delta_\gamma}$-null. Then there are Borel sets $\mathscr{S}'_\gamma \subseteq \mathscr{S}_\gamma$ such that

$$\forall \gamma \in \Gamma \ (\mathscr{S}'_\gamma, \gamma(\mathscr{S}_\gamma \setminus \mathscr{S}'_\gamma) \in \mathscr{I}).$$

Set $\mathscr{S} = [\mathscr{G}]^{<\infty} \setminus \bigcup_{\gamma \in \Gamma} \mathscr{S}'_\gamma \cup \gamma(\mathscr{S}_\gamma \setminus \mathscr{S}'_\gamma)$. Note that for all $S \in [\mathscr{G}]^{<\infty}$ and $\gamma \in \Gamma$, we have that either $S \in \mathscr{S}'_\gamma$, $\gamma \cdot S \in \gamma(\mathscr{S}_\gamma \setminus \mathscr{S}'_\gamma)$, or $S \cap \gamma \cdot S \neq \emptyset$, thus no pair of $\mathscr{E}$-related elements of $\mathscr{S}$ are disjoint. It follows from Proposition 7.3 of Kechris-Miller [4] that $\mathscr{E}|\mathscr{S}$ is smooth, thus $\mathscr{S} \in \mathscr{I}$, so $[\mathscr{G}]^{<\infty} \in \mathscr{I}$, which contradicts Proposition 3.4. ∎

Now fix $\gamma \in \Gamma$ such that $\mathscr{S}_\gamma \notin \mathscr{I}_{\Delta_\gamma}$, let T be the tree on $V = \Delta_\gamma$, and observe that $(\Delta_\gamma, \mathrm{id}, \mathscr{S}_\gamma)$ is a parameterized embedding of T into $\mathscr{G}$. ∎

A tree T on V is **non-linear** if some point of V has at least three T-neighbors.

PROPOSITION 5.3: *Suppose that there is no Borel way of selecting a point, end or line from each $\mathscr{G}$-component. Then there is a parameterized embedding of the non-linear tree on four points into $\mathscr{G}$.*

Proof. For each $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$, put $\Delta_{\gamma_1,\gamma_2,\gamma_3} = \{1_\Gamma, \gamma_1, \gamma_2, \gamma_3\}$ and $\partial\Delta_{\gamma_1,\gamma_2,\gamma_3} = \{\gamma_1, \gamma_2, \gamma_3\}$, and let $\mathscr{S}_{\gamma_1,\gamma_2,\gamma_3}$ consist of those $S \in [\mathscr{G}]^{<\infty}$ for which $S, \gamma_1 \cdot S, \gamma_2 \cdot S, \gamma_3 \cdot S$ are pairwise disjoint and the sets $\gamma_1 \cdot S, \gamma_2 \cdot S, \gamma_3 \cdot S$ lie in distinct $\mathscr{G}_{\hat{S}}$-components.

LEMMA 5.4: *There exist $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ such that $\mathscr{S}_{\gamma_1,\gamma_2,\gamma_3} \notin \mathscr{I}_{\partial\Delta_{\gamma_1,\gamma_2,\gamma_3}}$.*

Proof. Suppose, towards a contradiction, that each $\mathscr{S}_{\gamma_1,\gamma_2,\gamma_3}$ is $\mathscr{I}_{\partial\Delta_{\gamma_1,\gamma_2,\gamma_3}}$-null. Then there are Borel sets $\mathscr{S}_{\gamma_1,\gamma_2,\gamma_3,\delta}$, for $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ and $\delta \in \partial\Delta_{\gamma_1,\gamma_2,\gamma_3}$, such that for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$, the following conditions are satisfied:

1. $\mathscr{S}_{\gamma_1,\gamma_2,\gamma_3} = \bigcup_{\delta \in \partial\Delta_{\gamma_1,\gamma_2,\gamma_3}} \mathscr{S}_{\gamma_1,\gamma_2,\gamma_3,\delta}$.
2. $\forall \delta \in \partial\Delta_{\gamma_1,\gamma_2,\gamma_3} \ (\delta(\mathscr{S}_{\gamma_1,\gamma_2,\gamma_3,\delta}) \in \mathscr{I})$.

Set $\mathscr{S} = [\mathscr{G}]^{<\infty} \setminus \bigcup_{\gamma_1,\gamma_2,\gamma_3 \in \Gamma, \delta \in \partial\Delta_{\gamma_1,\gamma_2,\gamma_3}} \delta(\mathscr{S}_{\gamma_1,\gamma_2,\gamma_3,\delta})$. As in the proof of Lemma 5.2, the set $\mathscr{S}$ is linear, thus $[\mathscr{G}]^{<\infty} \in \mathscr{J}$, which contradicts Proposition 3.5. ∎

Now fix $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ such that $\mathscr{S}_{\gamma_1,\gamma_2,\gamma_3} \notin \mathscr{I}_{\Delta_{\gamma_1,\gamma_2,\gamma_3}}$, let T be the non-linear tree on $V = \Delta_{\gamma_1,\gamma_2,\gamma_3}$ centered at 1_Γ, and note that $(\Delta_{\gamma_1,\gamma_2,\gamma_3}, \mathrm{id}, \mathscr{S}_{\gamma_1,\gamma_2,\gamma_3})$ is a parameterized embedding of T into $\mathscr{G}$. ∎

Next, we use a similar argument to show that parameterized embeddings can always be extended to parameterized embeddings of larger trees. Given a one-step extension T' of T, we say that a parameterized embedding $(\Delta', \pi', \mathscr{S}')$ of T' into $\mathscr{G}$ **extends** $(\Delta, \pi, \mathscr{S})$ if there exists $\gamma \in \Gamma$ such that

$$\Delta' = \Delta \cup \Delta\gamma, \quad \pi'(wi) = \pi(w)\gamma^i \quad \text{and} \quad \mathscr{S}' \subseteq \mathscr{S} \cap \gamma^{-1}(\mathscr{S}).$$

In this case, we also say that $(\Delta', \pi', \mathscr{S}')$ is a **γ-extension** of $(\Delta, \pi, \mathscr{S})$.

We say that a zero-dimensional Polish topology τ on $[\mathscr{G}]^{<\infty}$ is **good** if it is compatible with the Borel structure which $[\mathscr{G}]^{<\infty}$ inherits from $X^{<\mathbb{N}}$, the group Γ acts on $[\mathscr{G}]^{<\infty}$ by τ-homeomorphisms, and each of the sets

$$\mathscr{S}_{\Delta,\delta,\gamma} = \{S \in [\mathscr{G}]^{<\infty} : S, \gamma \cdot S \text{ are } (\Delta,\delta)\text{-linkable}\}$$

is τ-clopen, where $\Delta \subseteq \Gamma$ is finite, $\delta \in \Delta$, and $\gamma \in \Gamma$. We say that a parameterized embedding $(\Delta, \pi, \mathscr{S})$ is **τ-continuous** if the set $\mathscr{S}$ is τ-clopen.

Proposition 5.5: *Suppose that τ is good and T is a finite tree with one-step extension T'. Then every τ-continuous parameterized embedding of T into $\mathscr{G}$ extends to a τ-continuous parameterized embedding of T' into $\mathscr{G}$.*

Proof. Suppose that $(\Delta, \pi, \mathscr{S})$ is a τ-continuous parameterized embedding of T into $\mathscr{G}$. Let V denote the vertex set of T, and fix $v \in V$ such that T' is the v-extension of T. For each $\gamma \in \Gamma$, set $\Delta_\gamma = \Delta \cup \Delta\gamma$, $\partial\Delta_\gamma = \pi(\partial T) \cup \pi(\partial T)\gamma$, and $\mathscr{S}_\gamma = \mathscr{S} \cap \gamma^{-1}(\mathscr{S}) \cap \mathscr{S}_{\Delta,\pi(v),\gamma}$.

Lemma 5.6: *There exists $\gamma \in \Gamma$ such that $\mathscr{S}_\gamma$ is $\mathscr{I}_{\partial\Delta_\gamma}$-positive.*

Proof. Suppose, towards a contradiction, that there are Borel sets $\mathscr{S}'_\gamma \subseteq \mathscr{S}_\gamma$ with

$$\forall \gamma \in \Gamma \ (\mathscr{S}'_\gamma, \gamma(\mathscr{S}_\gamma \setminus \mathscr{S}'_\gamma) \in \mathscr{I}_{\pi(\partial T)}).$$

Sublemma 5.7: *The set $\mathscr{S}' = \mathscr{S} \setminus \bigcup_{\gamma \in \Gamma} \mathscr{S}'_\gamma \cup \gamma(\mathscr{S}_\gamma \setminus \mathscr{S}'_\gamma)$ is $\mathscr{I}_{\pi(v)}$-null.*

Proof. By Sublemma 5.4 of Miller [5], there are Borel sets $\mathscr{S}_n \subseteq [\mathscr{G}]^{<\infty}$ such that each $\mathscr{S}_n$ is pairwise disjoint and $\mathscr{S}' = \bigcup_{n\in\mathbb{N}} \mathscr{S}_n$. For each $n \in \mathbb{N}$ and $S \in \mathscr{S}_n$, let $\Phi_n(\pi(v) \cdot S)$ be the $\mathscr{G}_{\widehat{\pi(v)\cdot S}}$-component which contains $\delta \cdot S$, for some (equivalently, all) $\delta \in \Delta \setminus \{\pi(v)\}$. It follows from the definition of $\mathscr{S}'$ that $\Phi_n \subseteq [\mathscr{G}]^{\rightarrow}$ is directed, thus Proposition 3.4 implies that $\pi(v) \cdot \mathscr{S}' = \bigcup_{n\in\mathbb{N}} \mathrm{dom}(\Phi_n)$ is directable, and the sublemma follows. ∎

It now follows that $\mathscr{S} \in \mathscr{I}_{\pi(\partial T)}$, the desired contradiction. ∎

Now fix $\gamma \in \Gamma$ such that $\mathscr{S}_\gamma$ is $\mathscr{I}_{\partial\Delta_\gamma}$-positive. Setting

$$\Delta' = \Delta_\gamma, \quad \pi'(wi) = \pi(w)\gamma^i \quad \text{and} \quad \mathscr{S}' = \mathscr{S}_\gamma,$$

it follows that $(\Delta', \pi', \mathscr{S}')$ is the desired extension of $(\Delta, \pi, \mathscr{S})$. ∎

Next, we use Proposition 5.5 to build parameterized embeddings of finite trees.

PROPOSITION 5.8: *Suppose that there is no Borel way of selecting a point or end from each $\mathscr{G}$-component. Then every finite linear tree admits a parameterized embedding into $\mathscr{G}$.*

Proof. As every finite linear tree embeds into a finite linear tree of cardinality 2^{n+1}, it is enough to prove the proposition for trees of this latter type. As all such trees are obtained via n one-step extensions of the tree on two points, this special case of the proposition therefore follows from Proposition 5.1 and n applications of Proposition 5.5. ∎

PROPOSITION 5.9: *Suppose that there is no Borel way of selecting a point, end or line from each $\mathscr{G}$-component. Then every finite tree admits a parameterized embedding into $\mathscr{G}$.*

Proof. Given a finite tree (T, V) and a set $W \subseteq V$, the **induced graph** on W is the set T_W of all pairs $(w_1, w_2) \in W \times W$ such that $w_1 \neq w_2$ and no point of W is strictly in-between w_1 and w_2. As every finite tree is isomorphic to an induced graph associated with a tree obtained through finitely many one-step extensions of the non-linear four point tree, the proposition follows from Proposition 5.3 and finitely many applications of Proposition 5.5. ∎

6. Building tail-to-end embeddings

Here we give the connection between parameterized and tail-to-end embeddings:

Proposition 6.1: *Suppose that $(T, V, s_0, s_1, \ldots)$ is an arboreal blueprint and there is a parameterized embedding of T into $\mathscr{G}$. Then there is a tail-to-end embedding of $\mathscr{T}$ into $\mathscr{G}$.*

Proof. Fix a parameterized embedding $(\Delta_0, \pi_0, \mathscr{S}_0)$ of T into $\mathscr{G}$, as well as an increasing sequence $\Gamma_0 \subseteq \Gamma_1 \subseteq \cdots$ of symmetric finite sets whose union is Γ. As in §2, we use T_n to denote the tree on $V \times 2^n$ associated with $(T, V, s_0, s_1, \ldots)$. Fix a good topology τ on $[\mathscr{G}]^{<\infty}$ with respect to which $(\Delta_0, \pi_0, \mathscr{S}_0)$ is continuous (the existence of such a topology follows, for example, from §13 of Kechris [3]). Fix also a countable clopen τ-basis $\mathscr{B}$.

For each $v \in V$, set $\delta_v = \pi_0(v)$. After replacing $\mathscr{S}_0$ by its intersection with an appropriate element of $\mathscr{B}$, we can assume that

$$\forall S \in \mathscr{S}_0 \, \forall \gamma \in \Gamma_0 \, \forall v, w \in V \ (\delta_w^{-1} \gamma \delta_v \cdot S \neq S \Rightarrow \delta_w^{-1} \gamma \delta_v \cdot S \notin \mathscr{S}_0).$$

We will recursively find clopen subsets $\mathscr{S}_1 \supseteq \mathscr{S}_2 \supseteq \cdots$ of $\mathscr{S}_0$ and elements $\gamma_1, \gamma_2, \ldots$ of Γ. Along the way, we will associate with each $n \geq 1$ the set

$$\Delta_n = \{\delta_s : s \in V \times 2^n\},$$

where $\delta_s \in \Gamma$ is given by

$$\delta_s = \delta_{s(0)} \gamma_1^{s(1)} \gamma_2^{s(2)} \cdots \gamma_n^{s(n)}.$$

We also define $\pi_n : V \times 2^n \to \Gamma$ by $\pi_n(s) = \delta_s$. All of this will be done in such a fashion that, for all $n \in \mathbb{N}$, the following conditions are satisfied:

1. $(\Delta_n, \pi_n, \mathscr{S}_n)$ is a parameterized embedding of T_n into $\mathscr{G}$;
2. if $n > 0$, then $\forall s, t \in V \times 2^{n-1} \, \forall \gamma \in \Gamma_{n-1} \ (\gamma \delta_s(\mathscr{S}_n) \cap \delta_t \gamma_n(\mathscr{S}_n) = \emptyset)$;
3. $\forall S \in \mathscr{S}_n \, \forall s, t \in V \times 2^n \, \forall \gamma \in \Gamma_n \ (\delta_t^{-1} \gamma \delta_s \cdot S \neq S \Rightarrow \delta_t^{-1} \gamma \delta_s \cdot S \notin \mathscr{S}_n)$;
4. $\forall s \in V \times 2^n \ (\mathrm{diam}(\delta_s(\mathscr{S}_n)) \leq 1/n)$.

Granting we have found $\mathscr{S}_i$ and γ_i, for $1 \leq i \leq n$, which satisfy (1)–(4), we must describe how to find γ_{n+1} and $\mathscr{S}_{n+1}$. By Proposition 5.5, there exists $\gamma_{n+1} \in \Gamma$ for which there is a γ_{n+1}-extension $(\Delta, \pi, \mathscr{S})$ of $(\Delta_n, \pi_n, \mathscr{S}_n)$. As $\gamma_{n+1}(\mathscr{S}) \subseteq \mathscr{S}_n$, condition (3) ensures that, for each $S \in \mathscr{S}$, we have that

$$\forall s, t \in V \times 2^n \, \forall \gamma \in \Gamma_n \ (\delta_t^{-1} \gamma \delta_s \cdot S \neq \gamma_{n+1} \cdot S).$$

It follows that there is a neighborhood $\mathscr{U} \in \mathscr{B}$ of S such that

(a) $\forall s,t \in V \times 2^n \, \forall \gamma \in \Gamma_n \; (\gamma\delta_s(\mathscr{U}) \cap \delta_t\gamma_{n+1}(\mathscr{U}) = \emptyset)$.

By further refining $\mathscr{U} \in \mathscr{B}$, we can ensure also that the following conditions hold:

(b) $\forall S' \in \mathscr{U} \, \forall s,t \in V \times 2^{n+1} \, \forall \gamma \in \Gamma_{n+1} \; (\delta_t^{-1}\gamma\delta_s \cdot S' \neq S' \Rightarrow \delta_t^{-1}\gamma\delta_s \cdot S' \notin \mathscr{U})$;

(c) $\forall s \in V \times 2^{n+1} \; (\mathrm{diam}(\delta_s(\mathscr{U})) \leq 1/(n+1))$.

It then follows that there exists $\mathscr{U} \in \mathscr{B}$ such that $\mathscr{S} \cap \mathscr{U} \notin \mathscr{I}_{\pi(\partial T_{n+1})}$. Set $\mathscr{S}_{n+1} = \mathscr{S} \cap \mathscr{U}$, and observe that $(\Delta_{n+1}, \pi_{n+1}, \mathscr{S}_{n+1})$ is a parameterized embedding of T_n into $\mathscr{G}$. This completes the description of γ_{n+1} and $\mathscr{S}_{n+1}$.

We are now ready to define the embedding. For each $n \in \mathbb{N}$ and $s \in V \times 2^n$, set $\mathscr{S}_s = \delta_s(\mathscr{S}_n)$, and define $\pi : V \times 2^{\mathbb{N}} \to [\mathscr{G}]^{<\infty}$ by

$$\pi(x) = \text{the unique element of } \bigcap_{n \in \mathbb{N}} \mathscr{S}_{x|n}.$$

Conditions (2) and (4) easily imply that π is a continuous injection.

Lemma 6.2: *Suppose that $(x,y) \notin F_{n+1}$. Then $\forall \gamma \in \Gamma_n \; (\gamma \cdot \pi(x) \neq \pi(y))$.*

Proof. Fix $m > n$ such that $x(m) \neq y(m)$. By switching the roles of x,y if necessary, we can assume that $x(m) = 0$ and $y(m) = 1$. Suppose, towards a contradiction, that there exists $\gamma \in \Gamma_n$ with $\gamma \cdot \pi(x) = \pi(y)$, and define $S_x, S_y \in \mathscr{S}_m$ by

$$S_x = \delta_{x|m}^{-1} \cdot \pi(x) \quad \text{and} \quad S_y = \gamma_m^{-1}\delta_{y|m}^{-1} \cdot \pi(y).$$

It follows that

$$\pi(y) = \gamma\delta_{x|m} \cdot S_x = \delta_{y|m}\gamma_m \cdot S_y,$$

which contradicts the fact that $\gamma\delta_{x|m}(\mathscr{S}_m) \cap \delta_{y|m}\gamma_m(\mathscr{S}_m) = \emptyset$. ∎

Corollary 6.3: *Suppose that $(x,y) \notin E_0$. Then $(\pi(x), \pi(y)) \notin E$.*

Next, we note that the construction of π ensures that there is a simple relationship between the images of E_0-related elements of $V \times 2^{\mathbb{N}}$:

Lemma 6.4: *Suppose that xF_ny. Then $\delta_{x|(n+1)}^{-1} \cdot \pi(x) = \delta_{y|(n+1)}^{-1} \cdot \pi(y)$.*

Proof. Simply observe that

$$\begin{aligned}\{\delta_{y|(n+1)}\delta^{-1}_{x|(n+1)}\cdot\pi(x)\} &= \delta_{y|(n+1)}\delta^{-1}_{x|(n+1)}\Big(\bigcap_{m\geq n}\mathscr{S}_{x|(m+1)}\Big)\\ &= \bigcap_{m\geq n}\delta_{y|(n+1)}\delta^{-1}_{x|(n+1)}(\mathscr{S}_{x|(m+1)})\\ &= \bigcap_{m\geq n}\mathscr{S}_{y|(m+1)}\\ &= \{\pi(y)\},\end{aligned}$$

thus $\delta^{-1}_{x|(n+1)}\cdot\pi(x)=\delta^{-1}_{y|(n+1)}\cdot\pi(y)$. ∎

Corollary 6.5: *π is an embedding of E_0 into $\mathscr{E}$.*

It still remains to check that

$$(x,y)\in\mathscr{T}\Leftrightarrow(\pi(x),\pi(y))\in\mathscr{G}_{\mathscr{S}},$$

for all $x,y\in V\times 2^{\mathbb{N}}$. By Corollary 6.5, we can assume that xE_0y, thus $\pi(x)\mathscr{E}\pi(y)$. Fix a $\mathscr{G}_{\mathscr{S}}$-path $\pi(x_0),\pi(x_1),\ldots,\pi(x_k)$ from $\pi(x)$ to $\pi(y)$ of minimal length, and find $n\in\mathbb{N}$ sufficiently large that $x_0F_nx_1F_n\cdots F_nx_k$. As $(\Delta_n,\pi_n,\mathscr{S}_n)$ is a parameterized embedding of T_n into $\mathscr{G}$, it follows that

$$(x,y)\in\mathscr{T}\;\Leftrightarrow\;(x|(n+1),y|(n+1))\in T_n\;\Leftrightarrow\;k=1\;\Leftrightarrow\;(\pi(x),\pi(y))\in\mathscr{G}_{\mathscr{S}},$$

which completes the proof of the proposition. ∎

7. The main results

Here we combine the results of the previous sections to obtain our dichotomies:

Theorem 7.1: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a graphing of E, and $(T,V,s_0,s_1,\ldots)$ is a linear arboreal blueprint. Then exactly one of the following holds:*

1. *there is a Borel way of selecting a point or end from each $\mathscr{G}$-component;*
2. *there is a tail-to-end embedding of $\mathscr{T}$ into $\mathscr{G}$.*

Proof. To see that (1) and (2) are mutually exclusive suppose, towards a contradiction, that there is a Borel way of selecting a point or end from each $\mathscr{G}$-component, and there is a tail-to-end embedding of $\mathscr{T}$ into $\mathscr{G}$. Proposition

4.1 then ensures that there is a Borel way of selecting a point or end from each $\mathscr{T}$-component, which contradicts Proposition 2.2.

It remains to check that $\neg(1) \Rightarrow (2)$. Suppose that there is no Borel way of selecting a point or end from each $\mathscr{G}$-component. It then follows from Proposition 5.8 that there is a parameterized embedding of T into $\mathscr{G}$, thus Proposition 6.1 ensures that there is a tail-to-end embedding of $\mathscr{T}$ into $\mathscr{G}$. ∎

THEOREM 7.2: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a graphing of E, and $(T, V, s_0, s_1, \ldots)$ is a non-linear arboreal blueprint. Then exactly one of the following holds:*

1. *There is a Borel way of selecting a point, end or line from each $\mathscr{G}$-component.*
2. *There is a tail-to-end embedding of $\mathscr{T}$ into $\mathscr{G}$.*

Proof. To see that (1) and (2) are mutually exclusive suppose, towards a contradiction, that there is a Borel way of selecting a point, end or line from each $\mathscr{G}$-component, and there is a tail-to-end embedding of $\mathscr{T}$ into $\mathscr{G}$. Proposition 4.1 then ensures that there is a Borel way of selecting a point, end or line from each $\mathscr{T}$-component, which contradicts Proposition 2.2.

It remains to check that $\neg(1) \Rightarrow (2)$. Suppose that there is no Borel way of selecting a point, end or line from each $\mathscr{G}$-component. It then follows from Proposition 5.9 that there is a parameterized embedding of T into $\mathscr{G}$, thus Proposition 6.1 ensures that there is a tail-to-end embedding of $\mathscr{T}$ into $\mathscr{G}$. ∎

As a corollary, we now have the following

THEOREM 7.3: *Suppose that X is a Polish space, E is a countable Borel equivalence relation, $\mathscr{G}$ is a graphing of E, and there is a Borel way of selecting a non-empty closed set of countably many ends from each $\mathscr{G}$-component. Then there is a Borel way of selecting an end or line from each $\mathscr{G}$-component.*

Proof. Suppose, towards a contradiction, that there is no Borel way of selecting an end or line from each $\mathscr{G}$-component. As every $\mathscr{G}$-component has an end, it follows that there is no Borel way of selecting a point, end or line from each $\mathscr{G}$-component. Fix a non-linear arboreal blueprint $(T, V, s_0, s_1, \ldots)$. Then Theorem 7.2 ensures that there is a tail-to-end embedding of $\mathscr{T}$ into $\mathscr{G}$, and Theorem 4.1 gives a Borel way of choosing a point or non-empty closed set of countably many ends from each $\mathscr{T}$-component, which contradicts Proposition 2.2. ∎

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References

[1] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Transactions of the American Mathematical Society **234** (1977), 289–324.

[2] S. Jackson, A. S. Kechris and A. Louveau, *Countable Borel equivalence relations*, Journal of Mathematical Logic **2** (2002), 1–80.

[3] A. S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.

[4] A. S. Kechris and B. D. Miller, *Topics in Orbit Equivalence*, Lecture Notes in Mathematics, vol. 1852, Springer-Verlag, Berlin, 2004.

[5] B. D. Miller, *Ends of graphed equivalence relations, I*, Israel Journal of mathematics (this volume).